ON THE GENERAL SOLUTION OF EQUATIONS OF AXISYMMETRICAL MOTIONS OF VISCOUS FLUIDS

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Some forms of the general solution of the equations of axisymmetrical steady-state slow motions of viscous fluids are considered. The general solution is constructed in a form which at the same time consitutes the system of basic formulas of the integration method based on the properties of p-analytic functions [1].

1. Let us consider the Stokes equations and the equation of continuity in cylindrical coordinates and in the case of axisymmetrical steady-state motions,

$$\frac{\partial p}{\partial r} - \mu\Delta_1 v_r + \mu\frac{v_r}{r^2} = 0 \qquad (1.1)$$

$$\frac{\partial p}{\partial z} - \mu\Delta_1 v_z = 0 \qquad \left(\Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) \qquad (1.2)$$

$$\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r} = 0 \qquad (1.3)$$

Eq. (1.3) indicates that there exists a stream function $\Psi(r, z)$ such that

$$v_r = \frac{1}{r}\frac{\partial \Psi}{\partial z}, \qquad v_z = -\frac{1}{r}\frac{\partial \Psi}{\partial r} \qquad (1.4)$$

Substituting these Expressions into (1.1) and (1.2), we find that Ψ satisfies Eq.

$$\Delta_2\Delta_2\Psi = 0 \qquad \left(\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{r}\frac{\partial}{\partial r}\right) \qquad (1.5)$$

2. In order to integrate Eq. (1.5) we make use of p-analytic functions. Let us recall their definitions and some of their properties which we shall have occasion to use below (e.g. see [1], Chapter 1, Sections 2 and 3). The function $f(\zeta) = u(x, y) + iv(x, y)$ of the complex variable $\zeta = x + iy$ is called p-analytic with the characteristic $p = p(x, y)$ in the domain D if it is single-valued in this domain and if its real and imaginary parts have continuous partial derivatives and satisfy the system of Eqs.

$$\frac{\partial u}{\partial x} = \frac{1}{p}\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{1}{p}\frac{\partial v}{\partial x} \qquad (2.1)$$

If $p = p(\beta)$, where β is a harmonic function of x and y and if $\omega = \alpha + i\beta$ is an analytic function of $\zeta = x + iy$, then by the operator derivative of the function $f(\zeta)$ with respect to the conjugate variable Z we mean the Expression (2.2)

$$\frac{d_p' f(\zeta)}{dZ} = \frac{1}{2}\left(\frac{\partial u}{\partial \alpha} + \frac{1}{p}\frac{\partial v}{\partial \beta}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial \alpha} - p\frac{\partial u}{\partial \beta}\right) \qquad \left(Z = X + iY = \alpha + i\int\frac{d\beta}{p}\right)$$

and by the operator derivative of the function $f(\zeta)$ with respect to the anticonjugate variable $\bar{Z} = X - iY$ the Expression

$$\frac{d_p' f(\zeta)}{d\bar{Z}} = \frac{1}{2}\left(\frac{\partial u}{\partial \alpha} - \frac{1}{p}\frac{\partial v}{\partial \beta}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial \alpha} + p\frac{\partial u}{\partial \beta}\right) \qquad (2.3)$$

If $f(\zeta)$ is a p-analytic function, then

$$\frac{d_p' f(\zeta)}{dZ} = \frac{\partial u}{\partial \alpha} + i\,\frac{\partial v}{\partial \alpha} = \frac{1}{p}\frac{\partial v}{\partial \beta} - ip\,\frac{\partial u}{\partial \beta}, \quad \frac{d_p' f(\zeta)}{d\overline{Z}} = 0 \tag{2.4}$$

The single-valued function of a complex variable $f(\zeta)$ is called operator-integrable over the conjugate variable Z in the domain D if there exists a function of a complex variable $f^*(\zeta)$ which is continuous in the domain D and such that $d_p' f^*(\zeta)/dZ$ exists and Eq. $d_p' f^*(\zeta)/dZ = f(\zeta)$ is valid in the domain D.

The indefinite operator integral over the conjugate variable Z of the function $f(\zeta)$ is written as $\int' f(\zeta)\, d_p' Z$.

In order for the operator integral over the conjugate variable Z of the arbitrary p-analytic function $f(\zeta)$ to be a p-analytic function it is necessary and sufficient that $p = p(\beta)$, where β is a harmonic function of x and y. Now let us attempt to find the general solution of the fourth-order Eqs.

$$\Delta_1^*\Delta_1^* u = 0 \qquad \left(\Delta_1^* = \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} + \frac{1}{p}\frac{\partial p}{\partial \beta}\frac{\partial}{\partial \beta}\right) \tag{2.5}$$

$$\Delta_2^*\Delta_2^* v = 0 \qquad \left(\Delta_2^* = \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} - \frac{1}{p}\frac{\partial p}{\partial \beta}\frac{\partial}{\partial \beta}\right) \tag{2.6}$$

If $f = u + iv$, then Eqs. (2.5) and (2.6) can be written as

$$\frac{d_p'^4 f(\zeta)}{dZ^2\, d\overline{Z}^2} = 0$$

From this, making use of the indefinite operator integral over the conjugate variable Z, we obtain the general solutions of these Eqs. in the form

$$u = 2\alpha\,[\Phi_1(\zeta) + \overline{\Phi}_1(\zeta)] + \chi(\zeta) + \overline{\chi}(\zeta) \tag{2.7}$$

$$v = 2\alpha\,[\Phi_1(\zeta) - \overline{\Phi}_1(\zeta)] + \chi(\zeta) - \overline{\chi}(\zeta) \tag{2.8}$$

where $\Phi_1(\zeta)$ and $\chi(\zeta)$ are arbitrary p-analytic functions and $\overline{\Phi}_1(\zeta)$ and $\overline{\chi}(\zeta)$ are the complex conjugates of the functions $\Phi_1(\zeta)$ and $\chi(\zeta)$, respectively.

3. Setting $\zeta = x + iy = r + iz$, $p = r^{-1}$, $\omega = \alpha + i\beta = -z + ir$, we reduce Eq. (2.5) to (1.5); hence, the general solution of Eq. (1.5) is of the form

$$\Psi = -2z\,[\Phi_1(\zeta) + \overline{\Phi}_1(\zeta)] + \chi(\zeta) + \overline{\chi}(\zeta) \tag{3.1}$$

where $\Phi_1(\zeta)$ and $\chi(\zeta)$ are arbitrary r^{-1}-analytic functions and $\overline{\Phi}_1(\zeta)$ and $\overline{\chi}(\zeta)$ are the corresponding conjugate functions. However, in order to simplify the formulas to follow, we use the following expression for Ψ:

$$\Psi = z\,[\Phi_1(\zeta) + \overline{\Phi}_1(\zeta)] + \chi(\zeta) + \overline{\chi}(\zeta) \tag{3.2}$$

which is equivalent to (3.1).

Writing $V_r = rv_r$, taking account of relations (1.4) and (2.3) and also of the fact that Ψ is a real function, and making use of Expression (3.2), we obtain

$$V_r + iv_z = \frac{\partial \Psi}{\partial z} - i\,\frac{1}{r}\frac{\partial \Psi}{\partial r} = -2\,\frac{d_p'\Psi}{d\overline{Z}} = \Phi_1(\zeta) - 2z\,\overline{\frac{d_p'\Phi_1(\zeta)}{dZ}} - \overline{\Phi}_2(\zeta) \tag{3.3}$$

where $\Phi_1(\zeta)$ and $\Phi_2(\zeta) = -\Phi_1(\zeta) + 2d_p'\chi(\zeta)/dZ$ are arbitrary r^{-1}-analytic functions of $\zeta = r + iz$. We introduce the notation

$$2\Omega = r\left(\frac{\partial v_z}{\partial r} - \frac{\partial v_r}{\partial z}\right) = r\,\frac{\partial v_z}{\partial r} - \frac{\partial V_r}{\partial z}$$

Taking account of Eq. (1.1) and Eq. (1.3) differentiated with respect to r, and then of Eq. (1.2) and Eq. (1.3) differentiated with respect to z, we have

$$\frac{\partial(2\mu\Omega)}{\partial r} = r\,\frac{\partial p}{\partial z}, \qquad \frac{\partial(2\mu\Omega)}{\partial z} = -r\,\frac{\partial p}{\partial r}$$

Hence, $2\mu\Omega + ip$ is an r^{-1}-analytic function of $\zeta = r + iz$. Making use of the continuity Eq. $(1/r)\,\partial V_r/\partial r + \partial v_z/\partial z = 0$, we obtain

$$2\Omega = \left(r\,\frac{\partial v_z}{\partial r} - \frac{\partial V_r}{\partial z}\right) - i\left(\frac{1}{r}\,\frac{\partial V_r}{\partial r} + \frac{\partial v_z}{\partial z}\right) = 2\,\frac{d_p'(V_r + iv_z)}{dZ} = 4R\left\{\frac{d_p'\Phi_1(\zeta)}{dZ}\right\}$$

Hence,

$$2\mu\Omega + ip = 4\mu\,\frac{d_p'\Phi_1(\zeta)}{dZ} \tag{3.4}$$

By virtue of relations (2.4), we can also write Formulas (3.3) and (3.4) as

$$V_r + iv_z = \Phi_1(\zeta) + 2z\,\frac{\partial\overline{\Phi}_1(\zeta)}{\partial z} - \overline{\Phi}_2(\zeta), \quad 2\mu\Omega + ip = -4\mu\,\frac{\partial\Phi_1(\zeta)}{\partial z} \tag{3.5}$$

Formula (3.5), which is the general solution of system (1.1) to (1.3), can also be transformed by replacing the complex variable $\zeta = r + iz$ by the complex variable $z = x + iy$ and the functions $-i\Phi_1(\zeta)$ and $-i\Phi_2(\zeta)$ by the functions $\Phi_1(z)$ and $\Phi_2(z)$, respectively.

But if the function $f(\zeta)$ is p-analytic, then the function $if(\zeta)$ is also p^{-1} analytic. This implies that Formulas (3.5) can be rewritten in the form

$$v_y - iV_x = \Phi_1(z) - 2y\frac{\partial\overline{\Phi}_1(z)}{\partial y} + \overline{\Phi}_2(z), \qquad p - 2i\mu\Omega = -4\mu\frac{\partial\Phi_1(z)}{\partial y} \qquad (V_x = xv_x) \quad (3.6)$$

Here $\Phi_1(z)$ and $\Phi_2(z)$ are arbitrary x-analytic functions of $z = x + iy$.

Formulas (3.6) are analogous to the basic formulas developed in [2 and 3] for the case of plane flows of viscous fluids. They constitute both the general solution of system (1.1) to (1.3) and the basic formulas for the application of p-analytic functions in viscous fluid hydrodynamics. Relations analogous to the first Formulas of (3.5) and (3.6) were obtained by Polozhii in elasticity theory [4].

4. Formulas (3.5) enable us to derive still other forms of the general solution of system (1.1) to (1.3). To this end we set

$$\Phi_1(\zeta) - 2z\frac{\partial\overline{\Phi}_1(\zeta)}{\partial z} + \overline{\Phi}_2(\zeta) = -r\frac{\partial\Lambda}{\partial r} - i\frac{\partial\Lambda}{\partial z} \qquad (4.1)$$

where $\Lambda(r, z)$ is a real function, and introduce the r^{-1}-analytic function

$$\Phi_1^*(\zeta) = M + iN = 2\int '\Phi_1(\zeta)d_p'Z$$

Here Z is the conjugate variable corresponding to the characteristic $p = r^{-1}$ and $\omega = \alpha + i\beta = -z + ir$. By (2.4) we have

$$2\Phi_1(\zeta) = \frac{d_p'\Phi_1^*(\zeta)}{dZ} = -\frac{\partial M}{\partial z} - i\frac{\partial N}{\partial z} = r\frac{\partial N}{\partial r} - i\frac{1}{r}\frac{\partial M}{\partial r} \qquad (4.2)$$

so that the first Formula of (3.5) can also be written as

$$V_r + iv_z = 2\Phi_1(\zeta) - \left[\Phi_1(\zeta) - 2z\frac{\partial\overline{\Phi}_1(\zeta)}{\partial z} + \overline{\Phi}_2(\zeta)\right] = r\frac{\partial}{\partial r}(\Lambda + N) + i\frac{\partial}{\partial z}(\Lambda - N) \qquad (4.3)$$

On the other hand, writing $\Phi_1(\zeta) = P + iQ$, $\Phi_2(\zeta) = R + iS$, we find from (4.1) that

$$r\frac{\partial\Lambda}{\partial r} = -P + 2z\frac{\partial P}{\partial z} - R, \qquad \frac{\partial\Lambda}{\partial z} = -Q - 2z\frac{\partial Q}{\partial z} + S$$

Hence, $\Delta_1\Lambda = -4\partial Q/\partial z = 2\partial^2 N/\partial z^2$. Taking account of this relation and also of continuity Eq. (1.3) written in the form $\Delta_1(\Lambda + N) - 2\partial^2 N/\partial z^2 = 0$, we find from Formula (4.3) that $\Delta_1 N = 0$. Since $\Delta_1\Lambda = 2\partial^2 N/\partial z^2$ and $\Delta_1 N = 0$, we have $\Delta_1\Delta_1\Lambda = 0$.

Finally, from the second relation of (3.5) we find that $p = -4\mu\partial Q/\partial z = \mu\Delta_1\Lambda$. Hence, the general solution of system (1.1) to (1.3) can be written as [5]

$$v_r = \frac{\partial}{\partial r}(\Lambda + N), \quad v_z = \frac{\partial}{\partial z}(\Lambda - N), \quad p = \mu\Delta_1\Lambda, \quad \Delta_1 N = 0, \quad \Delta_1\Delta_1\Lambda = 0 \quad (4.4)$$

which is analogous to Love's first form in the theory of elasticity ([6], p. 275).

5. Let us set $\Lambda + N = \partial\Xi/\partial z$ and $2\partial N/\partial z = \Delta_1\Xi$ in (4.4); since $\Delta_1 N = 0$, we have $\Delta_1\Delta_1\Xi = 0$. It follows that the general solution of system (1.1) to (1.3) can be written as [5]

$$v_r = \frac{\partial^2\Xi}{\partial r\,\partial z}, \quad v_z = -\Delta_1\Xi + \frac{\partial^2\Xi}{\partial z^2}, \quad p = \mu\frac{\partial}{\partial z}\Delta_1\Xi, \quad \Delta_1\Delta_1\Xi = 0 \qquad (5.1)$$

which is analogous to Love's second form in the theory of elasticity ([6], p. 276).

6. Let us set $\Lambda - N = \psi + \eta$ and $\partial N/\partial r = -\psi/r$ in (4.4); since $\Delta_1 N = 0$ and $\Delta_1\Lambda = 2\partial^2 N/\partial z^2$, we have $\Delta_1\eta = 0$ and $\Delta_2\varphi = 0$. The general solution of system (1.1) to (1.3) can therefore be written as [5]

$$v_r = -2\frac{\psi}{r} + \frac{\partial}{\partial r}(\psi + \eta), \quad v_z = \frac{\partial}{\partial z}(\psi + \eta), \quad p = \mu\Delta_1\psi, \quad \Delta_1\eta = 0, \ \Delta_2\psi = 0 \quad (6.1)$$

which is analogous to Timpe's formula in the theory of elasticity [7].

7. Let us set $\Lambda = (1/r)\partial^2\omega/\partial r\partial z$ in (4.4) and consider a function $T(r, z)$ such that the function $T + 2iN$ of the complex variable $\zeta = r + iz$ is an r^{-1}-analytic function. Since $\Delta_1 N = 0$, it follows that $\Delta_2 T = 0$, and since $\Delta_1\Lambda = 2\partial^2 N/\partial z^2$ and $\Delta_1[(1/r)\partial\omega/\partial r] = (1/r)\partial(\Delta_2\omega)/\partial r$, the mixed derivative $\partial^2(\Delta_2\omega - T)/\partial r\partial z = 0$. Hence we have $\Delta_2\omega = T + A(r) + B(z)$, where $A(r)$ and $B(z)$ are arbitrary functions. If we take $A(r) = 0$, $B(z) = 0$, we obtain the

general solution of system (1.1) to (1.3) in the form [5]

$$v_r = \frac{1}{2r}\frac{\partial T}{\partial z} - \frac{1}{r}\frac{\partial^3 \omega}{\partial z^3}, \qquad v_z = -\frac{1}{2r}\frac{\partial T}{\partial r} + \frac{1}{r}\frac{\partial^3 \omega}{\partial r\, \partial z^2}$$
$$p = \frac{\mu}{r}\frac{\partial^2}{\partial r\, \partial z}\Delta_2 \omega, \qquad \Delta_2 T = 0, \quad \Delta_2 \Delta_2 \omega = 0 \tag{7.1}$$

which is analogous to G.D. Grodskii's form in the theory of elasticity (e.g. see [8], Section 51).

Formulas (7.1) can also be written as

$$v_r = \frac{1}{r}\frac{\partial}{\partial z}\left(\frac{1}{2}\Delta_2 \omega - \frac{\partial^2 \omega}{\partial z^2}\right), \qquad v_z = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{2}\Delta_2 \omega - \frac{\partial^2 \omega}{\partial z^2}\right)$$
$$p = \frac{\mu}{r}\frac{\partial^2}{\partial r \partial z}\Delta_2 \omega, \qquad \Delta_2 \Delta_2 \omega = 0 \tag{7.2}$$

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